Real Composition Algebras

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Outline

Algebraic Structures

- Sets of elements (the objects)
- Operations
- Axioms (rules or properties)

Examples

- The properties of addition and mutliplication for real numbers (from basic algebra) form an algebraic structure called a field.
- The operation "wins" in the game paper-rock-scissors obeys the commutative laws, but is nonassociative:

$$pr = rp = p, \quad ps = sp = s, \quad rs = sr = r$$
$$p(rs) = pr = p, \quad but \quad (pr)\,s = ps = s$$
$$p\,(rs) \neq (pr)\,s$$

Algebraic Structures in Composition Algebras

Sets of elements (the objects)

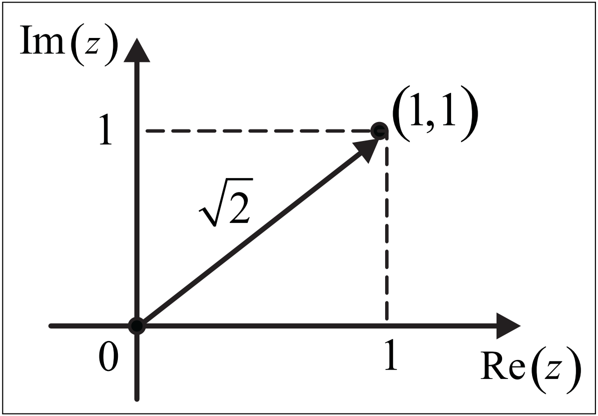
- Real numbers $x = (x_1)$
- Complex numbers $x = (x_1, x_2)$
- Hypercomplex numbers $x = (x_1, x_2, \ldots, x_n)$

Operations

- Vector addition
- Vector multiplication
- Scalar multiplication

We will address the properties after we talk about the elements.

Why should pairs of numbers also be numbers?



Number Systems

Different kinds of numbers

- whole numbers, natural numbers, integers
- rational numbers, irrational numbers, algebraic numbers, real numbers
- complex numbers, hypercomplex numbers

Vector Space

our first structure

- Intuitively, a **vector** is a quantity that has a magnitude (distance) and a direction.
- When we combine all the directions we can go with all the possible distances, we get what we call a **vector space.**
- A vector space comes with a way to combine two vectors called **addition.**
- Also, a vector can be multiplied by a scalar. (multiplication as repeated addition)

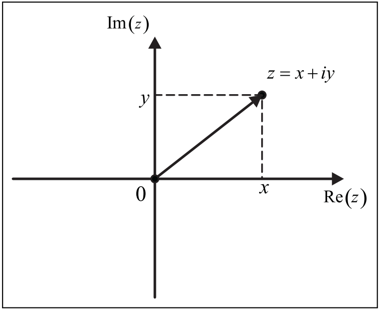
Substructures in an Algebra

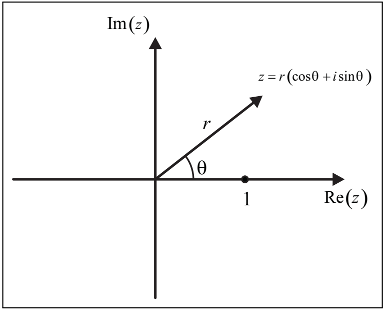
An algebra is a vector space with a way to multiply vectors in the space.

An **algebra** is built from

- A vector space which is built from
  - A set of scalars (the real numbers for real algebras)
    Scalars are from a field.
  - A set of vectors (for our purpose, the n-tuples)
  - Operations
    - Vector addition $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$
      Vectors form an Abelian group under addition
    - Scalar multiplication $\lambda x = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$
- Vector multiplication is a **bilinear binary operation**

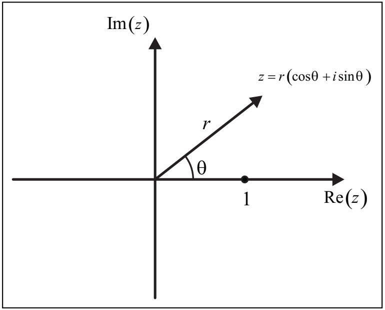
Composing Rotation

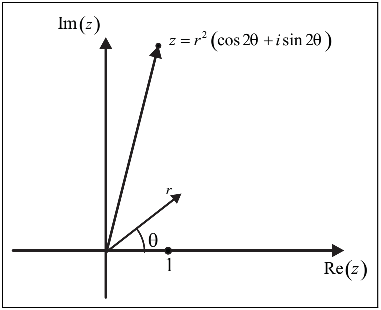




Here are two ways to represent a vector.

Composing Rotation





Multiplication multiplies the length and composes rotation.

But that is not why it is called a composition algebra

Composing bilinear forms is the motivation.

While investigating the representability of natural numbers by quadratic forms, Gauss (Disq. Arith. Art. 235) defined the binary quadratic form of $N(x) = ax_1^2 + 2bx_1x_2 + cx_2^2$ to be **the composition of quadratic forms** $P(y)$ and $Q(z)$ if $N(x) = P(y)Q(y)$ holds for all y and all z for some bilinear form with **integer** coefficients. He then showed that a linear change of variables converts the form into the two squares problem:

$$\left(x_1^2 + x_2^2\right) = \left(u_1^2 + u_2^2\right)\left(v_1^2 + v_2^2\right)$$

Composing bilinear forms is the motivation.

Hurwitz defined **the composition of quadratic forms** in n-dimensions and showed that $N(x) = P(y)\,Q(y)$ can be converted into the sum of squares

$$\left(x_1^2 + x_2^2 + \cdots + x_n^2\right) = \left(u_1^2 + u_2^2 + \cdots + u_n^2\right)\left(v_1^2 + v_2^2 + \cdots + v_n^2\right)$$

without loss of generality.
We will use a norm $[x] = \sum x_i^2 = xIx^T$ with the standard inner product $2[x, y] = [x + y] - [x] - [y]$.

Defining multiplication.

Real composition algebra

We will call an algebra a **real composition algebra** if it has a unit and if the product of the norms of any two elements composes the norm of their product

$$[x]\,[y] = [xy]$$

.

Complex Multiplication

$$(x_1, x_2)\,(y_1, y_2) = (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1)$$

There is no 3 dimensional composition algebra

LEGENDRE counterexample (1830)

We have $3 = 1^2 + 1^2 + 1^2$ and $21 = 4^2 + 2^2 + 1^2$ but $3 \cdot 21 = 63$ is not the sum of three squares.

The Dickson doubling procedure will show us that the dimension of a composition algebra must always be a power of 2.

Dickson Doubling Procedure

Dickson double algebra

Suppose Y is a composition algebra and X is a proper subalgebra with unit. Since $\dim(X) < \dim(Y)$ for a proper subalgebra, there is a basis i_Y in Y that is orthogonal to X. We call the algebra $Y = X + i_Y X$ the Dickson double algebra of X.

Conway and Baez prove that iX is an orthogonal complement to X algebraically. A double algebra has twice the dimension of the algebra it doubles. If is a proper subalgebra, we can continue doubling until no orthogonal vectors remain. Thus, we know that $2^n \dim(X) = \dim(Y)$.

The product of a double algebra

Hypercomplex Arithmetic

Conway uses facts about the inner product with $N(xy) = N(x)\,N(y)$ to show

$$(a + i_Z b)(c + i_Z d) = \left(ac - d\bar{b}\right) + i_Z\left(cb + \bar{a}d\right)$$

for any $a, b, c, d \in Y$ with where i_Z is orthogonal to Y.

Outline of the proof

Lemma

When a composition algebra Z contains a subalgebra Y it must also contain its Dickson double.

Outline of the proof

Lemma

Z is a composition algebra only when it is the double of an associative composition algebra Y.

Sketch of proof

Conway shows the property $N(xy) = N(x)N(y)$ holds for $x = a + i_Z b$ and $y = c + i_Z d$ with $a, b, c, d \in Y$ holds when $(ac)b = a(cb)$.

Outline of the proof

Lemma

Z is a composition algebra only when it is the double of an associative composition algebra Y.

Two more lemmas show that doubling can survive at most three Dickson doublings.

Outline of the proof

Lemma

Z is a composition algebra only when it is the double of an associative composition algebra Y.

Lemma

Y is an associative composition algebra only when it is the double of an associative, commutative composition algebra X.

Lemma

X is an associative, commutative composition algebra only when it is the double of an associative, commutative composition algebra with trivial conjugation.

Outline of the proof

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Hurwitz's Theorem (1898)

Theorem

The real numbers, complex numbers, quaternions, and octonions are the only composition algebras

Proof from Conway's lemmas.

The real numbers from a commutative, associative composition algebra. The complex numbers are the associative, commutative double of the real numbers. The quaternions are the associative, noncommutative double of the complex numbers, since complex numbers have notrivial conjugation. The octonions are the nonassociative double of the quaternions. And, the Dickson double of octonions are no longer a composition algebra. □

Results for Composition algebras

Weierstrass (1863)

Proves the complex numbers are the only commutative, associative composition algebra (applying the fundamental theorem of algebra)

Frobenius (1877)

Proves any associative real composition algebra is isomorphic to either the real numbers, the complex numbers, or the quaternions.

Results for Composition algebras

Hopf (1940)

Proves every commutative real composition algebra is at most two dimensional.
Proves Hopf's Lemma, which has the Fundamental Theorem of Algebra and the Gelfand-Mazur Theorem as "simple" corollaries (Ebbinghaus et al.)